Interference Alignment for the K User MIMO Interference Channel

Akbar Ghasemi, Abolfazl Seyed Motahari, and Amir Keyvan Khandani

Coding & Signal Transmission Laboratory (www.cst.uwaterloo.ca)

Department of Electrical and Computer Engineering, University of Waterloo

Waterloo, ON, Canada N2L3G1

{aghasemi,abolfazl,khandani}@cst.uwaterloo.ca

Abstract

The K user multiple input multiple output (MIMO) Gaussian interference channel with M antennas at each transmitter and N antennas at each receiver is considered. We assume that channel coefficients are fixed and are available at all transmitters and receivers. The main objective of this paper is to characterize the total number of degrees of freedom (DoF) for this channel. We show that for *fixed channel coefficients* $\frac{MN}{M+N}K$ degrees of freedom can be achieved. The achivability method is based on a new technique for interference alignment recently advised by Motahari *et al.* [17]. Also we provide a new upper-bound on the total number of DoF for this channel. This upper-bound coincide with our achievable DoF for $K \geq \frac{M+N}{gcd(M,N)}$ where $gcd(M,N)$ denotes the greatest common divisor of M and N. Because there is no cooperation among transmit and/or receive antennas of each user in our approach, our results are applicable to cellular systems in which a base station with multiple antennas communicates with several users each with single antenna. For this case, as the number of users in each cell increases, the total number of DoF also increases and approaches to the interference free DoF.

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I. Introduction

INTERFERENCE MANAGEMENT is one of the biggest challenges in wireless networks, in which multiple transmissions occur concurrently over a common communication medium. Interference is usually handled in practice by either *interference avoidance* in which users coordinate their transmissions by orthogonalizing their signals in time or frequency, or by *treating-interference-as-noise* in which users increase their transmission power and treat each other's interference as noise. *Interference decoding*, although it is less practical, is another approach to deal with interference when interference is strong enough to be decoded along with the desired signal.

During the past three decades, information theorists have made extensive efforts to characterize the ultimate obstruction that interference impose on the capacity of wireless networks. For the two-user Gaussian interference channel (GIFC) case, which has received the most attention to date, the capacity region is completely known for some ranges of channel coefficients [1]- [6]. For the general two-user case, the best result is that of [7] in which the approximate capacity region is found to within one bit.

By moving from two-user case to *more than two users*, the capacity characterization even become more challenging. In fact, in order to reduce the sever effects of interference for $K > 2$ users, the use of a new technique known as *interference alignment* is essential. Interference alignment which was firstly introduced by Maddah-Ali *et. al* [8], [9] in the context of MIMO X channels, is an elegant technique that practically reduces the effect of the aggregated interference from several users to the effect of the interference from one user only. There are two versions of interference alignment in literature: *signal space alignment* and *signal scale alignment*. In signal space alignment methods, as it originally proposed in [8], by deploying some linear pre-coding at transmitters and combining operations at receivers, the interference channel practically converted into multiple non-interfering Gaussian channels. Signal space alignment approaches are applicable to interference channels with time varying/frequency selective channel coefficients. Signal scale alignment schemes, on the other hand, use structured coding, e.g. lattice codes, to align interference in the signal level and are especially useful in the case of constant channels. In [10], for the special cases of many-to-one and one-to-many GIFC, authors computed the capacity region within constant bits using the signal scale alignment. For the general $K > 2$ users GIFC, most of the effort has focused on a simpler problem: what is the total number of degrees of freedom (DoF) for a $K > 2$ user GIFC? The total number of DoF for a GIFC shows the growth of the maximum achievable sum rates in the limit of increasing signal to noise ratio ($\mathtt{SNR}$). Using the idea of signal space interference alignment, Cadambe and Jafar in [11] showed that for a fully connected K user GIFC with time varying/frequency selective channel coefficients,

the total number of DoF is equal to $\frac{K}{2}$, i.e. each user can enjoy half of its available number of DoF in spite of interfering signals from other users. For the constant channel case, Host-Madsen and Nosratinia in [12], conjectured that the total number of DoF is upper-bounded by one regardless of the number of users. Etkin and Ordentlich in [13] deployed some results of additive combinatorics to show that for a constant fully connected real GIFC, the total number of DoF is very sensitive to the rationality/irrationality of channel coefficients. They showed that for a fully connected constant real GIFC with rational channel coefficients, the total number of DoF is strictly less than $\frac{K}{2}$. Moreover, they showed that for a class of measure zero of channel coefficients, the total number of DoF is equal to $\frac{K}{2}$. Independently, Motahari *et. al* showed in [14] that for a three user constant symmetric real GIFC with irrational channel coefficients, the total number of DoF is equal to $\frac{3}{2}$. However, their assumption on the structure of channel, i.e. symmetric channel, restricted its scope to a subset of measure zero of all possible channel coefficients. For a constant GIFC with complex channel coefficients, Cadambe *et. al* in [15], settled the Host-Madsen and Nosratinia conjecture in the negative by introducing asymmetric complex signaling. They showed that the K user complex GIFC with constant coefficients has at least 1.2 degrees of freedom for almost all values of channel coefficients. Recently, Motahari *et. al* settled the problem in general by proposing a new type of interference alignment that can achieve $\frac{K}{2}$ degrees of freedom for almost all K-user real GIFC with constant coefficients [16], [17]. The key tools introduced in [17] to establish this result is a new technique of interference alignment which used results from Diophantine approximation in Number theory [18] to show that interference can be aligned based on the properties of rationals and irrationals.

Extending the aforementioned results to the K user multiple input multiple output (MIMO) interference channel is straightforward when the number of transmit antennas is equal to the number of receive antennas. In fact, based on the results of [11], [17], it is not hard to see that for a K user $M \times M$ MIMO interference channel the total number of DoF is equal to $\frac{KM}{2}$ whether the channel is constant or time verying/frequnecy selective. However, extending this conclusion to the general K user $M \times N$ MIMO interference channel is not straightforward. In [19], the authors proved that for the K user $M \times N$ MIMO interference channel, the total number of degrees of freedom is equal to $K \min(M, N)$ if $K \leq R$ where $R = \lfloor \frac{\max(M,N)}{\min(M,N)} \rfloor$ and $\lfloor \cdot \rfloor$ is the floor function. This result holds for both time-varying and constant channel coefficients. For $K > R$, they showed that $\frac{R}{R+1} \min(M, N)K$ degrees of freedom can be achieved if channel coefficients are time-varying and drawn from a continuous distribution. They also proved that for $K > R$, the total number of DoF is upper-bounded by $\frac{\max(M,N)}{R+1}K$ whether the channel is constant or time-varying. Another related work is [20] in which Suh and Tse considered the problem of interference

alignment for cellular networks. Using a method called subspace interference alignment, they showed that for a cellular system as the number of users in each cell increases, the total number of DoF also increases and approaches to the interference free DoF. To show this surprising result, they introduced the randomness of multi-path channels into their equations.

In this paper, we extend the results of [19] in two directions. First, we show that their results can be extended to *constant* channels by generalizing the method of [17] to the MIMO case. Second, we improve their results by introducing a better achievable DoF and a tighter upper-bound.

This paper is organized as follows. In section II, the system model is introduced. In section III, the main results of this paper is presented and some discussions are followed. In section IV, we present a new outer-bound on the total number of DoF for a MIMO GIFC. In section V, we introduce preliminaries on interference alignment and summarize some relevant results that will be used in this paper. In section VI, we demonstrate our achievability method for a three user 1×2 MIMO GIFC. In section VII, we generalize the results of section VI to the K user $M \times N$ MIMO GIFC. We conclude in section VIII.

Notation: $\mathbb{Z}$ and $\mathbb{Z}^+$ represent the set of integers and positive integers respectively. $(a, b)_{\mathbb{Z}}$ denotes the set of integers between a and b. For a set S and a number a, we define the set $a.S$ as $a.S := \{a.s : s \in S\}$. Also for the two sets S_1 and S_2, the set theoretic difference would be denoted by $S_1 \setminus S_2 = \{s \in S_1 : s \notin S_2\}$. The union of two sets S_1 and S_2 will be denoted by $S_1 \bigcup S_2$. For two positive integer x and y, $gcd(x, y)$ denotes the greatest common devisor of x and y. In addition we use the following notations

$$\mathcal{K} = \{1, \cdots, K\}, \quad \mathcal{N} = \{1, \cdots, N\}, \quad \mathcal{M} = \{1, \cdots, M\}$$

II. System Model

We consider the time-invariant real K-user MIMO Gaussian interference channel. This channel is used to model a communication network with K transmitter-receiver pairs in which each transmitter, which is equipped with M antennas, tries to communicate to its corresponding receiver, which is equipped with N antennas. All transmitters share a common bandwidth and wish to have reliable communication at their maximum rates. The channel output at the k^{th} receiver is characterized by the following input-output relationship

$$\mathbf{Y}_k = \mathbf{H}_{k1}\mathbf{X}_1 + \mathbf{H}_{k2}\mathbf{X}_2 + \cdots + \mathbf{H}_{kK}\mathbf{X}_K + \mathbf{Z}_k$$

where, $k \in \mathcal{K}$ is the user index, $\mathbf{Y}_k = (Y_{k1}, \cdots, Y_{kN})^t$ is the $N \times 1$ output signal vector of the k^{th} receiver, $\mathbf{X}_j = (X_{j1}, \cdots, X_{jM})^t$ is the $M \times 1$ input signal vector of the j^{th} transmitter, $\mathbf{H}_{kj} = [H_{kj}(n, m)]$ is the $N \times M$ channel matrix between transmitter j and receiver k with the (n, m)-th entry specifying the

channel gain from m^{th} antenna of transmitter j to n^{th} antenna of receiver k and $\mathbf{Z}_k = (Z_{k1}, \cdots, Z_{kN})^t$ is $N \times 1$ additive white Gaussian noise (AWGN) vector at the k^{th} receiver. We assume all noise terms are i.i.d zero mean real Gaussian with unit variance. The total power across all transmitters is assumed to be equal to ρ. Hereafter, ρ will be referred to as $\mathtt{SNR}$.

A K-tuple of rates $\mathbf{R}(\rho) = (R_1(\rho), \cdots, R_K(\rho))$ is said to be achievable for a MIMO GIFC if the transmitters can increase the cardinalities of their message sets as $2^{nR_i(\rho)}$ with block length n and the probability of error for all messages can be simultaneously made arbitrary small by taking n sufficiently large. The capacity region $\mathcal{C}(\rho)$ of the K user MIMO GIFC is the set of all achievable K-tuples of rates $\mathbf{R}(\rho)$.

Our primary objective in this paper is to characterize the total number of DoF of the K-user MIMO GIFC. Let $\mathcal{C}(\rho)$ denote the capacity region of this channel. The DoF region associated with this channel is the shape of $\mathcal{C}(\rho)$ in high $\mathtt{SNR}$ regime scaled by $\log(\rho)$. If $\mathcal{R}$ denotes the DoF region of the MIMO GIFC, all extreme points of $\mathcal{R}$ can be obtained by solving the following optimization problem:

$$D_{\boldsymbol{\lambda}} = \lim_{\rho \to \infty} \max_{\mathbf{R}(\rho) \in \mathcal{C}(\rho)} \frac{\boldsymbol{\lambda}^t \mathbf{R}(\rho)}{\log(\rho)}. \tag{1}$$

The total number of DoF refers to the case of $\boldsymbol{\lambda} = (1, 1, \ldots, 1)$, i.e., it represents the behavior of maximum achievable sum rate as $\mathtt{SNR}$ goes to infinity. Throughout this paper, D denotes the total number of DoF of the system.

III. Main Result and discussions

The main results of this paper are the following two theorems.

Theorem 1: For the K user MIMO GIFC with M antennas at each transmitter and N antennas at each receiver and for real and constant channel coefficients, $\frac{MN}{M+N}K$ degrees of freedom can be achieved for *almost all* channels.

Theorem 2: For the K user MIMO GIFC with M antennas at each transmitter and N antennas at each receiver and for real and constant channel coefficients, the total number of DoF is bounded above by $\frac{MN}{M+N}K$ if $K \geq \frac{M+N}{gcd(M,N)}$. If $K < \frac{M+N}{gcd(M,N)}$, the total number of DoF is bounded above by $\min(D^-, D^+)$ where

$$D^- = \min_{\mu \in \mathbb{Z}^+} \frac{MNL_\mu^- + \mu \min(M,N) gcd(M,N)}{(M+N)L_\mu^- + \mu\, gcd(M,N)} K, \tag{2}$$

$$D^+ = \min_{\mu \in \mathbb{Z}^+} \frac{MNL_\mu^+ + \mu \max(M,N) gcd(M,N)}{(M+N)L_\mu^+ + \mu\, gcd(M,N)} K, \tag{3}$$

where L_μ^- and L_μ^+ are respectively the maximum element of the sets S_μ^- and S_μ^+ which are defined as

$$\begin{aligned} S_\mu^- = \big\{ L_{min} : & \max(M,N)L_{min} = \min(M,N)L_{max} - gcd(M,N)\mu \\ & L_{min} + L_{max} \leq K \\ & 0 \leq L_{min} \leq L_{max} \big\}, \end{aligned} \tag{4}$$

$$\begin{aligned} S_\mu^+ = \big\{ L_{max} : & \max(M,N)L_{min} = \min(M,N)L_{max} + gcd(M,N)\mu \\ & L_{min} + L_{max} \leq K \\ & 0 \leq L_{min} \leq L_{max} \big\}. \end{aligned} \tag{5}$$

we will set $L_\mu^- = 0$ ($L_\mu^+ = 0$) if S_μ^- (S_μ^+) is an empty set.

Remark 1: As it can be seen from Theorem 1 and 2, for $K \geq \frac{M+N}{gcd(M,N)}$, the total number of DoF is equal to $\frac{MN}{M+N}K$. For $K \leq \lfloor \frac{\max(M,N)}{\min(M,N)} \rfloor$, Gou and Jafar in [19] showed that the total number of DoF is equal to $K\min(M,N)$. While there is a complete characterization of DoF for $K \geq \frac{M+N}{gcd(M,N)}$ and $K \leq \lfloor \frac{\max(M,N)}{\min(M,N)} \rfloor$, this characterization for the case of $\lfloor \frac{\max(M,N)}{\min(M,N)} \rfloor < K < \frac{M+N}{gcd(M,N)}$ seems to be challenging. Our achievability scheme works poorly for this range. We are currently working to fill this gap.

Remark 2: It is not hard to see that if $\mu > \frac{\min(M,N)}{gcd(M,N)}K$, the set S_μ^- would be an empty set. Similarly, if $\mu > \frac{\max(M,N)-\min(M,N)}{2gcd(M,N)}K$, the set S_μ^+ would be an empty set. Thus, the minimization in (2) and (3) can be easily handled.

A. *Almost All versus All Cases*

In the statement of the Theorem 1, it is emphasized that the result is valid for *almost all* cases. It means that the collection of all possible H for them $\frac{MN}{M+N}K$ DoF is achievable has measure one. In other words, if all channel gains are drawn independently from a random distribution then almost surely all of them are irrational and satisfy properties required for achieving this DoF.

B. *Time varying versus Constant Channels*

Gou and Jafar in their work [19] proved that for a K user $M \times N$ MIMO interference channel with time varying channel coefficients $\min(M,N)\frac{R}{R+1}K$ DoF can be achieved if $K > R$ where $R = \lfloor \frac{\max(M,N)}{\min(M,N)} \rfloor$. In this paper, we improve this result in two directions. First, our achievable DoF is strictly greater than their achievable DoF when R in not an integer. Second, we remove the unrealistic condition of time varying channel coefficients. In fact, in our scheme the channel can be static over time and still it is possible to achieve this improved DoF for almost all channel realizations.

C. Cooperation Does not help

As we shall see shortly, there is no cooperation among transmit and/or receive antennas of each user in our achievability technique. Since for $K \geq \frac{M+N}{gcd(M,N)}$ our method is DoF optimal, we can conclude that when the number of interferers in the system are above a threshold, which depends on the number of transmit and receive antennas, cooperation provides no benefit from DoF point of view.

D. Implications on Cellular Systems

Consider a cellular system in which different cells are working in the proximity of each other. In each cell, there is a base station which is equipped with multiple antennas and covers all users in that cell. Since there is no cooperation among receive antennas in our approach, we can consider each cell as a MIMO transmit/receive pair in our MIMO interference network model where the number of transmit antennas is equal to the number of antennas of the base station and the number of receive antennas is equal to the number of users in that cell. When there is a large number of cells in the system, by increasing the number of users in each cell, the DoF per cell increases and approaches to the number of antennas of the base station. This means that each cell can achieve its interference free DoF in spite of several interfering cells in the system. This surprising result has been previously reported in [20].

IV. Outerbound on the DoF for the K user MIMO interference channel

In this section, we prove Theorem 2 which provides a new upper-bound on the total number of DoF of the K user MIMO interference channel.

Consider an L user MIMO GIFC with M antenna at each transmitter and N at each receiver where $L \leq K$ is a constant. Now, if we distribute these L users between two disjoint sets with L_1 and L_2 users ($L = L_1 + L_2$) and allow full cooperation among transmitters in each set and their corresponding receivers, then it is equivalent to the two user MIMO GIFC with L_1M, L_2M antennas at transmitters and L_1N, L_2N antennas at their corresponding receivers. It is proved in [21] that for a two user MIMO GIFC with M_1, M_2 antennas at transmitter 1, 2 and N_1, N_2 antennas at their corresponding receivers, the total number of DoF is equal to

$$J(M_1, M_2, N_1, N_2) = \min\{M_1 + M_2, N_1 + N_2, \max(M_1, N_2), \max(M_2, N_1)\}. \tag{6}$$

Since cooperation does not hurt the capacity, the DoF of the original L user interference channel does not exceed $J(L_1M, L_2M, L_1N, L_2N)$. Thus for any $i_1, i_2, \cdots, i_L \in \mathcal{K}$, $i_1 \neq i_2 \neq \cdots \neq i_L$ we have

$$d_{i_1} + d_{i_2} + \cdots + d_{i_L} \leq J(L_1M, L_2M, L_1N, L_2N), \tag{7}$$

where d_k denotes individual DoF achieved by user k. Since we can choose our L users among K users in $\binom{K}{L}$ different ways and each user k appears in exactly $\binom{K-1}{L-1}$ of these ways, by adding up all inequalities like (7), the total number of DoF of the K users GIFC is upper-bounded by

$$D \leq \frac{\binom{K}{L}}{\binom{K-1}{L-1}} J(L_1M, L_2M, L_1N, L_2N) = \frac{K}{L} J(L_1M, L_2M, L_1N, L_2N). \tag{8}$$

It is not hard to see that the function $J(L_1M, L_2M, L_1N, L_2N)$ can be upper-bounded as

$$J(L_1M, L_2M, L_1N, L_2N) \leq \max\{\max(M,N)L_{min}, \min(M,N)L_{max}\}, \tag{9}$$

where $L_{max} = \max(L_1, L_2)$ and $L_{min} = \min(L_1, L_2)$. We want to minimize the upper-bound in (8) over L, L_1, and L_2. The upper-bound in (8) would be minimized if we can possibly have

$$\max(M,N)L_{min} = \min(M,N)L_{max}. \tag{10}$$

However, there are some cases where (10) has no solution. we consider two cases:

1) $K \geq \frac{M+N}{gcd(M,N)}$

   In this case, we can satisfy (10) by choosing $L_{max} = \frac{\max(M,N)}{gcd(M,N)}$ and $L_{min} = \frac{\min(M,N)}{gcd(M,N)}$. Using these selections, (8) can be simplified as

$$D \leq \frac{J(L_1M, L_2M, L_1N, L_2N)}{L_{min} + L_{max}} K \leq \frac{\min(M,N)\max(M,N)}{\min(M,N) + \max(M,N)} K = \frac{MN}{M+N} K, \tag{11}$$

   which is the desired result.

2) $K < \frac{M+N}{gcd(M,N)}$

   In this case, (10) could not be satisfied and the the counterpart of (10) for this case would be

$$\max(M,N)L_{min} = \min(M,N)L_{max} \pm gcd(M,N)\mu, \tag{12}$$

   where μ is a positive integer that will be determined shortly. First consider that (12) is satisfied by minus sign. In this case the upper-bound in (8) can be simplified to

$$D \leq \frac{\min(M,N)L_{max}}{L_{min} + L_{max}} K = \frac{MNL_{min} + \mu \min(M,N) gcd(M,N)}{(M+N)L_{min} + \mu\, gcd(M,N)} K. \tag{13}$$

   Now, we should minimize the upper-bound in (13) over μ and L_{min} where μ can be any positive integer and L_{min} is the solution of the indeterminate equation (12) with minus sign subject to the constraints $0 \leq L_{min} \leq L_{max}$ and $L_{min} + L_{max} \leq K$. For each μ, let S_μ^- denote the set of all

solutions of (12) with minus sign under these constraints, i.e.

$$S_\mu^- = \big\{L_{min} : \max(M,N)L_{min} = \min(M,N)L_{max} - gcd(M,N)\mu$$

$$L_{min} + L_{max} \leq K$$

$$0 \leq L_{min} \leq L_{max}\big\}. \quad (14)$$

Note that for a fixed μ, the upper-bound in (13) is a decreasing function of L_{min}. Thus, for each μ in order to minimize the upper-bound over L_{min}, it is enough to consider the largest element of S_μ^-. If L_μ^- denotes the largest element of the set S_μ^-, the optimum upper-bound would be equal to

$$D^- = \min_{\mu\in\mathbb{Z}^+} \frac{MNL_\mu^- + \mu \min(M,N)gcd(M,N)}{(M+N)L_\mu^- + \mu\, gcd(M,N)}K. \quad (15)$$

For consistency, we set $L_\mu^- = 0$ whenever S_μ^- is an empty set. This does not loose the upper-bound because for every μ and L_{min}, we have $\frac{MNL_{min}+\mu \min(M,N)gcd(M,N)}{(M+N)L_{min}+\mu\, gcd(M,N)} < \min(M,N)$.

By repeating the same procedure for the case that (12) is satisfied by plus sign, the following upper-bound is obtained

$$D^+ = \min_{\mu\in\mathbb{Z}^+} \frac{MNL_\mu^+ + \mu \max(M,N)gcd(M,N)}{(M+N)L_\mu^+ + \mu\, gcd(M,N)}K, \quad (16)$$

where L_μ^+ denotes the largest element of the set S_μ^+ which is defined as

$$S_\mu^+ = \big\{L_{max} : \max(M,N)L_{min} = \min(M,N)L_{max} + gcd(M,N)\mu$$

$$L_{min} + L_{max} \leq K$$

$$0 \leq L_{min} \leq L_{max}\big\}. \quad (17)$$

Since (12) can be satisfied by both plus and minus signs, the final upper-bound would be equal to $\min(D^-, D^+)$. This completes the proof.

In order to illustrate how the upper-bound for $K < \frac{M+N}{gcd(M,N)}$ works, we give an example for $M = 5$, $N = 2$, and $K = 4$. We set $\mu = 1$ and looking for the largest possible value for L_{min} that simultaneously satisfies the following relations

$$5L_{min} = 2L_{max} - 1$$

$$0 \leq L_{min} \leq L_{max}$$

$$L_{min} + L_{max} \leq 4.$$

For $L_{min} = 2$, we can't find an L_{max} for that $5L_{min} = 2L_{max} - 1$. For $L_{min} = 1$, the only value of L_{max} that satisfies $5L_{min} = 2L_{max} - 1$ is $L_{max} = 3$ for that $L_{min} + L_{max} \leq 4$. Thus we have $L_1^- = 1$.

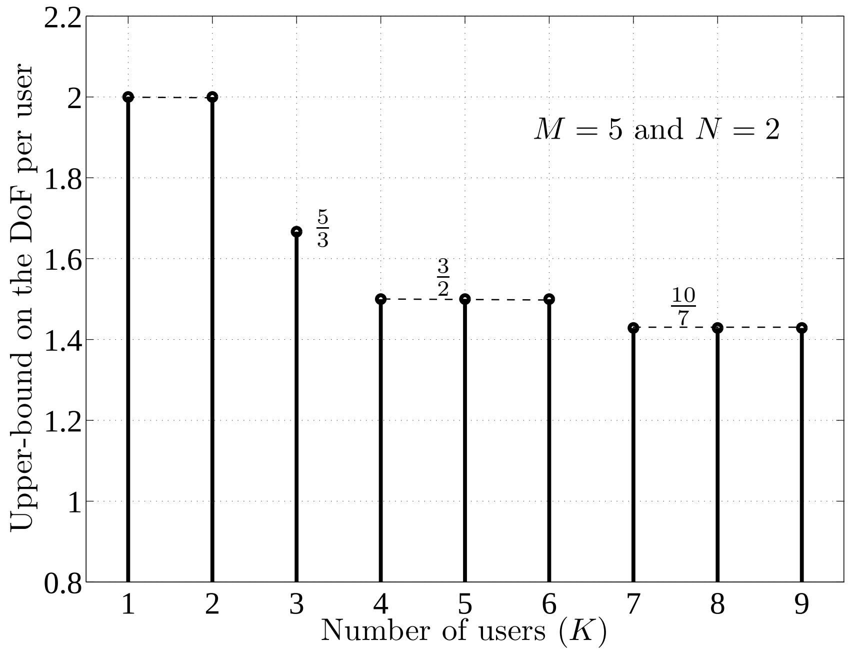

Fig. 1. Upper-bound on the DoF per user for $M = 5$ and $N = 2$

It is easy to check that $L_\mu^- = 0$ for $\mu \geq 2$. Substituting $\mu = 1$ and $L_1^- = 1$ in D^- yields $\frac{3}{2}$ as the upper bound. Fig. 1 depicts the result for $M = 5$ and $N = 2$ for different number of users.

As another example consider the case when $R = \frac{\max(M,N)}{\min(M,N)}$ is an integer. In this case $gcd(M, N) = \min(M, N)$ and hence $\frac{M+N}{gcd(M,N)} = R + 1$. Therefore, according to Theorem 2, for $K > R$ the total number of DoF is bounded above by $\frac{R}{R+1} \min(M, N)K$. For $K \leq R$, (12) is converted to $RL_{min} = L_{max} \pm \mu$ which for $\mu = 1$ yields $L_1^+ = R - 1$. Substituting these values in (16) yields $\min(M, N)K$ as an upper-bound. This upper-bound have been previously reported in [19].

V. A New Type of Interference Alignment

A new method for interference alignment has been recently introduced in a work by Motahari, Gharan, and Khandani [17]. By applying arguments from the field of Diophantine approximation in Number Theory, they showed that interference alignment can be performed based on the properties of rational and irrational numbers. They proved that using this new method of interference alignment, the total DoF of the K user *constant* GIFC with single antenna can be achieved. Since our results are based on this new method of interference alignment, we review their work here.

For a K user GIFC with single antenna, let $\mathbf{H} = [h_{ij}]$ denote the channel matrix with the (i, j)-th entry h_{ij} specifying the channel gain from transmitter j to receiver i. Let $\mathcal{H}$ denote the channel coefficient set,

that is the set of all channel gains, i.e.

$$\mathcal{H} = \{h_{11}, \cdots, h_{1K}, h_{21}, \cdots, h_{2K}, \cdots, h_{K1}, \cdots, h_{KK}\}.$$

Denote by $\Psi(\mathcal{H})$ the set of all monomials with variables from the set $\mathcal{H}$. That is a function ψ belongs to $\Psi(\mathcal{H})$ if it has a representation as $\psi = \prod_{i,j=1}^{K} h_{ij}^{s_{ij}}$, where s_{ij} are some non-negative integers. A function ψ belongs to $\Psi(\mathcal{H})$ will be denoted by $\psi(\mathcal{H})$.

We have the following theorem.

Theorem 3 ([17]): Let $\epsilon > 0$ be an arbitrary positive constant. For all $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^m$, and for any collection of m distinct functions $\psi_1(\mathcal{H}), \cdots, \psi_m(\mathcal{H})$ where $\psi_i(\mathcal{H}) \neq 1$, $i = 1, \cdots, m$, there is a constant κ such that the following inequality holds for almost all $\mathcal{H}$

$$|p + q_1\psi_1(\mathcal{H}) + q_2\psi_2(\mathcal{H}) + \cdots + q_m\psi_m(\mathcal{H})| > \frac{\kappa}{(\max_i |q_i|)^{m+\epsilon}}. \tag{18}$$

Due to the special structure of the elements of $\Psi(\mathcal{H})$, when an element of this set is multiplied by a channel coefficient, the result would have the same structure, i.e. it would be a monomial. Thus, as an immediate consequence of Theorem 3, if the transmitted signals of all users are integer linear combinations of the elements of set $\Psi(\mathcal{H})$, the minimum distance among different received signals will be bounded from below according to (18). Hence, the elements of set $\Psi(\mathcal{H})$ may be regarded as directions in the interference alignment context.

Now, we assume that the transmit signal of user k is an integer linear combination of some elements of $\Psi(\mathcal{H})$

$$X_k = A \sum_{l=0}^{L_k-1} u_{kl} T_{kl}, \tag{19}$$

where $u_{kl} \in (-Q, Q)_{\mathbb{Z}}$, $l = 0, \cdots, L_k - 1$ carry information of user k, T_{kl}, $l = 0, \cdots, L_k - 1$ are L_k distinct elements of $\Psi(\mathcal{H})$, and A controls the input power of all users. The set of all directions that are assigned to user k will be denoted by $\mathcal{T}_k$, i.e. $T_{kl} \in \mathcal{T}_k$, $l = 0, \cdots, L_k - 1$. The received signal at receiver j can be written as

$$Y_j = A(\sum_{l=0}^{L_j-1} h_{jj} T_{jl} u_{jl} + \underbrace{\sum_{k \in \mathcal{K}\setminus j} \sum_{l=1}^{L_k-1} h_{jk} T_{kl} u_{kl}}_{I_j}) + Z_j, \tag{20}$$

where I_j is the sum of interference of all other users at receiver j. The directions that exist in Y_j are again elements of $\Psi(\mathcal{H})$, but they are not necessarily distinct. In fact the goal of interference alignment in the present context is to reduce the number of distinct directions in interference part I_j. In other words,

I_j may be written as

$$I_j = \sum_{l=0}^{L'_j-1} u'_{jl} T'_{jl}, \tag{21}$$

where T'_{jl}, $l = 0, \cdots, L'_j - 1$ are L'_j distinct directions available in I_j ($L'_j \leq \sum_{k\in\mathcal{K}\setminus j} L_k$) and u'_{jl} is the sum of data arriving at direction T'_{jl}. The set of all distinct directions in I_j will be denoted by $\mathcal{T}'_j$. If the sets $h_{jj}\mathcal{T}_j$ and $\mathcal{T}'_j$ have no intersection, then in the absence of noise, receiver can decode all intended data streams error free and hence a multiplexing gain of $\frac{L_j}{L_j+L'_j}$ is achievable. The following theorem states this result more precisely.

Theorem 4 ([17]): Consider a K user GIFC with channel coefficient set $\mathcal{H}$ and assume each transmitter send its data according to (19). Let $\mathcal{T}_k$ denote the set of directions used by transmitter k and $\mathcal{T}'_k$ denote the set of distinct directions observed in the interference part of the received signal at receiver k. Moreover, assume that the components of $h_{kk}\mathcal{T}_k$ and $\mathcal{T}'_k$ are all distinct for any $k \in \mathcal{K}$. Then the following DoF is achievable for almost all channels $\mathcal{H}$

$$D = \frac{L_1 + L_2 + \cdots + L_K}{m+1}, \tag{22}$$

where $m = \max_{k\in\mathcal{K}} L_k + L'_k$ is the maximum number of directions in the received signal among all receivers.

The challenging part of this new interference alignment method is the selection of transmit directions for different users. In fact, if someone choose these directions randomly, the number of distinct directions in any received signal can be as large as $\sum_{k\in\mathcal{K}} L_k$ which yields the DoF equal to one. In the following, we see that how a clever design of transmit directions can limit the number of distinct directions in the received signal of user k to approximately $2L_k$ for any $k \in \mathcal{K}$.

In [17], authors selected the transmit directions for each user by the following procedure. Let Γ be a fixed positive integer. A direction T would be assigned as a transmit direction to user k if it can be represented as

$$T = \prod_{(i,j)\in\mathcal{K}\times\mathcal{K}} h_{ij}^{s_{ij}}, \tag{23}$$

where s_{ij} are integers whose ranges depend on k and Γ

$$\begin{aligned} s_{ii} &= 0 && (\forall i \in \mathcal{K}) \\ 0 \leq &s_{ik} \leq \Gamma - 1 && (\forall i \in \mathcal{K}\setminus\{k\}) \\ 0 \leq &s_{ij} \leq \Gamma && (\forall i \in \mathcal{K}), (\forall j \in \mathcal{K}\setminus\{k, i\}). \end{aligned} \tag{24}$$

The cardinality of the set $\mathcal{T}_k$, the set of all transmit directions assigned to user k, is given by

$$L_k = \Gamma^{K-1}(\Gamma+1)^{(K-1)^2}. \tag{25}$$

To compute the cardinality of the set $\mathcal{T}'_k$, the set of all distinct directions available in the interference part of the received signal of user k, it is enough to notice that these directions have a format like (23) except that the range of the exponents s_{ij} could change. Denote by $\mathcal{T}_r$ the set of all directions like (23) with exponents in the following range

$$\begin{aligned} s_{ii} &= 0 && (\forall i \in \mathcal{K}) \\ 0 \leq & s_{ij} \leq \Gamma && (\forall i \in \mathcal{K}), (\forall j \in \mathcal{K} \setminus \{i\}). \end{aligned} \tag{26}$$

It is not hard to see that for any $k \in \mathcal{K}$, the set $\mathcal{T}'_k$ is a subset of $\mathcal{T}_r$. Therefore, the cardinality of $\mathcal{T}'_k$ is given by

$$L'_k = (\Gamma+1)^{K(K-1)}. \tag{27}$$

Moreover, since elements of $\mathcal{T}_r$ do not have a h_{kk} factor, $\forall k \in \mathcal{K}$, the elements of $h_{kk}\mathcal{T}_k$ and $\mathcal{T}'_k$ are all distinct for any $k \in \mathcal{K}$. Hence, according to Theorem 4, the following DoF is achievable

$$D = \frac{K\Gamma^{K-1}(\Gamma+1)^{(K-1)^2}}{1+\Gamma^{K-1}(\Gamma+1)^{(K-1)^2}+(\Gamma+1)^{K(K-1)}} = \frac{K}{1+(1+\frac{1}{\Gamma})^{(K-1)}+\frac{1}{\Gamma^{K-1}(\Gamma+1)^{1+(K-1)^2}}}.$$

Since Γ is an arbitrary constant, taking the supremum over Γ yields that the total number of DoF of the K user GIFC with single antenna is equal to $\frac{K}{2}$.

VI. Proof and discussion of achievability method for a 3 user 1×2 interference channel

In this section, we explain our achievability method for a 3 user 1×2 MIMO GIFC. As we will see later, extending the proof to the general case is straightforward.

Our achievability result is based on the new interference alignment scheme explained in the previous section. The key idea here is how to extend this method to the multi antenna cases. Once again, the challenging part is the selection of transmit directions for different users. Consider the 3 user MIMO GIFC depicted in Fig. 2 in which direct paths are denoted by bold lines and interference paths are

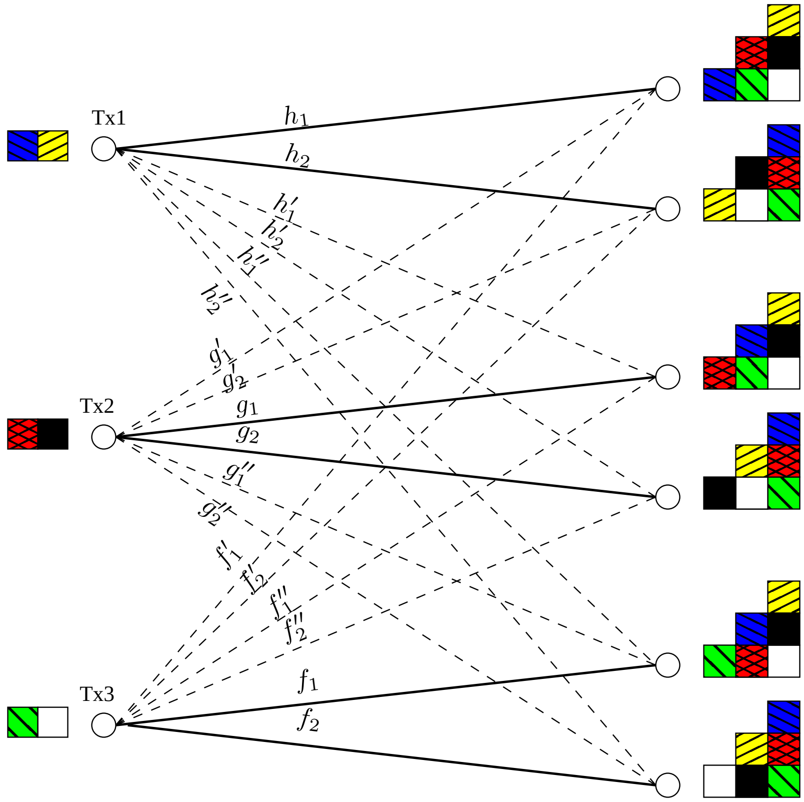

Fig. 2. Interference alignment for a 3 user 1×2 interference channel

denoted by dashed lines. The input-output relationship of this channel can be written as

$$\begin{aligned} Y_{11} &= h_1 X_1 + g_1' X_2 + f_1' X_3 + Z_{11}, \\ Y_{12} &= h_2 X_1 + g_2' X_2 + f_2' X_3 + Z_{11}, \\ Y_{21} &= h_1' X_1 + g_1 X_2 + f_1'' X_3 + Z_{21}, \\ Y_{22} &= h_2' X_1 + g_2 X_2 + f_2'' X_3 + Z_{22}, \\ Y_{31} &= h_1'' X_1 + g_1'' X_2 + f_1 X_3 + Z_{31}, \\ Y_{32} &= h_2'' X_1 + g_2'' X_2 + f_2 X_3 + Z_{32}. \end{aligned} \tag{28}$$

We select the transmit signal for different users as

$$\begin{aligned} X_1 &= h_1 X_1(1) + h_2 X_1(2), \\ X_2 &= g_1 X_2(1) + g_2 X_2(2), \\ X_3 &= f_1 X_3(1) + f_2 X_3(2), \end{aligned} \tag{29}$$

where $X_k(1)$ is intended to be decodable at the first receive antenna of user k and $X_k(2)$ is intended to

be decodable at the second receive antenna of user k for $k = 1, 2, 3$. Substituting (29) in (28), we have

$$\begin{aligned}
Y_{11} &= h_1^2 X_1(1) + \underbrace{g_1 g_1' X_2(1) + f_1 f_1' X_3(1)} + \underbrace{h_1 h_2 X_1(2) + g_1' g_2 X_2(2) + f_1' f_2 X_3(2)} + Z_{11},\\
Y_{12} &= \underbrace{h_1 h_2 X_1(1) + g_1 g_2' X_2(1) + f_1 f_2' X_3(1)} + h_2^2 X_1(2) + \underbrace{g_2' g_2 X_2(2) + f_2' f_2 X_3(2)} + Z_{12},\\
Y_{21} &= g_1^2 X_2(1) + \underbrace{h_1 h_1' X_1(1) + f_1 f_1'' X_3(1)} + \underbrace{h_1' h_2 X_1(2) + g_1 g_2 X_2(2) + f_1'' f_2 X_3(2)} + Z_{21},\\
Y_{22} &= \underbrace{h_1 h_2' X_1(1) + g_1 g_2 X_2(1) + f_1 f_2'' X_3(1)} + g_2^2 X_2(2) + \underbrace{h_2 h_2' X_1(2) + f_2 f_2'' X_3(2)} + Z_{22},\\
Y_{31} &= f_1^2 X_3(1) + \underbrace{h_1 h_1'' X_1(1) + g_1 g_1'' X_2(1)} + \underbrace{h_1'' h_2 X_1(2) + g_1'' g_2 X_2(2) + f_1 f_2 X_3(2)} + Z_{31},\\
Y_{32} &= \underbrace{h_1 h_2'' X_1(1) + g_1 g_2'' X_2(1) + f_1 f_2 X_3(1)} + f_2^2 X_3(2) + \underbrace{h_2 h_2'' X_1(2) + g_2 g_2'' X_2(2)} + Z_{32}.
\end{aligned} \tag{30}$$

As it can be seen from (30), channel coefficients appear in pairs in the input-output relationship. This suggests that we should choose the transmit directions as monomial with variables from these pairs. We deliberately write the input-output relationship in the form of (30) to separate the effects of the first and second part of each transmit signal on the output signals. The goal is to align the signals with a brace under them such that the effect of their sum acts like each of them from the DoF point of view. To do so, like the single antenna case, we select $X_k(1), X_k(2)$, $k \in \mathcal{K}$ as

$$X_k(1) = \sum_{l=0}^{L_k(1)-1} u_{kl}(1) T_{kl}(1),$$

$$X_k(2) = \sum_{l=0}^{L_k(2)-1} u_{kl}(2) T_{kl}(2),$$

where like single antenna case $u_{kl}(1) \in \mathcal{U} = (-Q, Q)_{\mathbb{Z}}$, $l = 0, \cdots, L_k(1) - 1$ and $u_{kl}(2) \in \mathcal{U} = (-Q, Q)_{\mathbb{Z}}$, $l = 0, \cdots, L_k(2) - 1$. $T_{kl}(1) \in \mathcal{T}_k(1)$ where $\mathcal{T}_k(1)$ is the set of all directions assigned to $X_k(1)$. Similarly $T_{kl}(2) \in \mathcal{T}_k(2)$. The problem is how to select $\mathcal{T}_k(1)$ and $\mathcal{T}_k(2)$ for different k's. We define three sets H, G, and F as follows

$$H = \{h_1, h_2, h_1', h_2', h_1'', h_2''\},$$

$$G = \{g_1, g_2, g_1', g_2', g_1'', g_2''\},$$

$$F = \{f_1, f_2, f_1', f_2', f_1'', f_2''\}.$$

we select directions for the first and second part of transmit signal k respectively as

$$\prod_w \zeta_{wk}(1)^{s_{wk}(1)}, \tag{31}$$

$$\prod_w \zeta_{wk}(2)^{s_{wk}(2)}. \tag{32}$$

where

$$\zeta_{wk}(1) \in (h_1.H \setminus \{h_1\}) \bigcup (g_1.G \setminus \{g_1\}) \bigcup (f_1.F \setminus \{f_1\}),$$
$$\zeta_{wk}(2) \in (h_2.H \setminus \{h_2\}) \bigcup (g_2.G \setminus \{g_2\}) \bigcup (f_2.F \setminus \{f_2\}). \tag{33}$$

Assume Γ is a fixed positive integer. The exponents $s_{wk}(1)$, $s_{wk}(2)$ are selected by the following rules

$$\begin{aligned}
0 \leq & s_{w1}(1) \leq \Gamma - 1 && \text{if } \zeta_{w1}(1) \in (h_1.H \setminus \{h_1\}) \\
0 \leq & s_{w1}(1) \leq \Gamma && \text{if } \zeta_{w1}(1) \in (g_1.G \setminus \{g_1\}) \bigcup (f_1.F \setminus \{f_1\}) \\
0 \leq & s_{w2}(1) \leq \Gamma - 1 && \text{if } \zeta_{w2}(1) \in (g_1.G \setminus \{g_1\}) \\
0 \leq & s_{w2}(1) \leq \Gamma && \text{if } \zeta_{w2}(1) \in (h_1.H \setminus \{h_1\}) \bigcup (f_1.F \setminus \{f_1\}) \\
0 \leq & s_{w3}(1) \leq \Gamma - 1 && \text{if } \zeta_{w3}(1) \in (f_1.H \setminus \{f_1\}) \\
0 \leq & s_{w3}(1) \leq \Gamma && \text{if } \zeta_{w3}(1) \in (h_1.H \setminus \{h_1\}) \bigcup (g_1.G \setminus \{g_1\})
\end{aligned} \tag{34}$$

$$\begin{aligned}
0 \leq & s_{w1}(2) \leq \Gamma - 1 && \text{if } \zeta_{w1}(2) \in (h_2.H \setminus \{h_2\}) \\
0 \leq & s_{w1}(2) \leq \Gamma && \text{if } \zeta_{w1}(2) \in (g_2.G \setminus \{g_2\}) \bigcup (f_2.F \setminus \{f_2\}) \\
0 \leq & s_{w2}(2) \leq \Gamma - 1 && \text{if } \zeta_{w2}(2) \in (g_2.G \setminus \{g_2\}) \\
0 \leq & s_{w2}(2) \leq \Gamma && \text{if } \zeta_{w2}(2) \in (h_2.H \setminus \{h_2\}) \bigcup (f_2.F \setminus \{f_2\}) \\
0 \leq & s_{w3}(2) \leq \Gamma - 1 && \text{if } \zeta_{w3}(2) \in (f_2.H \setminus \{f_2\}) \\
0 \leq & s_{w3}(2) \leq \Gamma && \text{if } \zeta_{w3}(2) \in (h_2.H \setminus \{h_2\}) \bigcup (g_2.G \setminus \{g_2\}).
\end{aligned} \tag{35}$$

Using these selections, we claim that all signals in (30) with a brace under them are aligned. To see this, look at the coefficients of $X_1(1)$ in the first part of the received signals in (30). Thanks to our selections, all channel pairs appear in the first part of the received signals have been used in $X_1(1)$ with an exponent up to Γ except those belong to $h_1.H \setminus \{h_1\}$ that used with an exponent up to $\Gamma - 1$. Since $X_1(1)$ has all the elements of $h_1.H \setminus \{h_1\}$ as its coefficients in the first part of the output signals, we conclude that the exponents of any element in $X_1(1)$ would not exceed Γ in the first part of the output signals. The same story is true for all other signals. In fact, if we define a set $\mathcal{T}_r(1)$ as the set of all directions like (31) with $0 \leq s_{wk}(1) \leq \Gamma$ for all $k \in \mathcal{K}$, the set of all directions in any interference term in the first part of the received signals would be a subset of $\mathcal{T}_r(1)$. Fig. 2 depicts the effect of this alignment on the received signals. The number of distinct directions in $X_k(1)$ and $X_k(2)$ is equal to

$$L_k(1) = L_k(2) = \Gamma^{2\times 3-1}(\Gamma+1)^{2\times(2\times 3-1)} = \Gamma^5(\Gamma+1)^{10}. \tag{36}$$

The number of distinct interference directions in any of the received signals, e.g. in Y_{11} is

$$L_1'(1) = 2(\Gamma+1)^{15}. \tag{37}$$

Hence using arguments similar to what explained in the previous section and according to Theorem 4, we can conclude that the following number of DoF is achievable per receive antenna

$$D = \frac{\Gamma^5(\Gamma+1)^{10}}{1+\Gamma^5(\Gamma+1)^{10}+2\times(\Gamma+1)^{15}}. \tag{38}$$

Since Γ is an arbitrary constant, taking supremum over Γ yield DoF equal to $\frac{1}{3}$ per receive antenna which leads to a total number of DoF equal to $3\times 2\times\frac{1}{3}=2$.

VII. General Case: K user $M \times N$ MIMO interference channel

In this section, we show that how the results of the previous section can be easily generalized to the K user $M\times N$ interference channel.

We choose the transmitted signal of user k on antenna m to have the following form

$$X_{km} = H_{kk}(1,m)\,X_{km}(1) + \cdots + H_{kk}(N,m)\,X_{km}(N), \quad \forall m \in \mathcal{M}, \tag{39}$$

where $X_{km}(n)$, $n\in\mathcal{N}$ is designed to be decodable at the n^{th} receive antenna of user k and seems like interference for the other receive antennas of this user. The received signal at the n^{th} received antenna of user k can be written as

$$Y_{kn} = \sum_{m\in\mathcal{M}} H_{kk}(n,m)X_{km} + \sum_{\substack{j\in\mathcal{K}\setminus k\\ m\in\mathcal{M}}} H_{kj}(n,m)X_{jm} + Z_{kn}, \quad \forall n\in\mathcal{N}. \tag{40}$$

Substituating (39) in (40) yields

$$\begin{aligned} Y_{kn} = &\sum_{m\in\mathcal{M}} H_{kk}^2(n,m)\,X_{km}(n) \\ &+ \sum_{\substack{m\in\mathcal{M}\\ n'\in\mathcal{N}\setminus n}} H_{kk}(n,m)H_{kk}(n',m)\,X_{km}(n') \\ &+ \sum_{\substack{j\in\mathcal{K}\setminus k\\ m\in\mathcal{M}\\ n'\in\mathcal{N}}} H_{kj}(n,m)H_{jj}(n',m)\,X_{jm}(n'). \end{aligned} \tag{41}$$

Similar to the 3 user case, we should carefully distribute the available degrees of freedom among $X_{km}(n)$, $n\in\mathcal{N}$. We select $X_{km}(n)$ as

$$X_{km}(n) = \sum_{l=0}^{L_{k,m}(n)-1} u_{kml}(n)T_{kml}(n), \tag{42}$$

where $u_{kml}(n) \in \mathcal{U} = (-Q, Q)_{\mathbb{Z}}$, $l = 0, \cdots, L_{k,m}(n) - 1$ carry information and $T_{kml}(n) \in \mathcal{T}_{km}(n)$ is a constant real number playing the role of a direction in our interference alignment scheme. $\mathcal{T}_{km}(n)$ is the set of directions assigned to $X_{km}(n)$ and $L_{k,m}(n)$ denotes its cardinality. Each of the elements of $\mathcal{T}_{km}(n)$ is in the following form

$$\prod_{\substack{(i,j)\in\mathcal{K}\times\mathcal{K} \\ (m',n')\in\mathcal{M}\times\mathcal{N} \\ n'\neq n \text{ if } j=i}} \left[H_{jj}(n,m')H_{ij}(n',m')\right]^{s_{j,m',n',i}}, \tag{43}$$

where $s_{j,m',n',i}$'s are integers taking values in the following ranges depends on the values of k, m, and n

$$\begin{aligned} 0 \leq & s_{k,m,n',k} \leq \Gamma - 1 && (\forall n' \in \mathcal{N} \setminus \{n\}) \\ 0 \leq & s_{k,m,n',i} \leq \Gamma - 1 && (\forall n' \in \mathcal{N}), (\forall i \in \mathcal{K} \setminus \{k\}) \\ 0 \leq & s_{j,m',n',i} \leq \Gamma && (\forall j \in \mathcal{K} \setminus \{k\}), (\forall m' \in \mathcal{M}), (\forall n' \in \mathcal{N}), (\forall i \in \mathcal{K}). \end{aligned} \tag{44}$$

In order to calculate the multiplexing gain of the system, we need to determine the cardinality of $\mathcal{T}_{km}(n)$. From (44) we have

$$L_{k,m}(n) := |\mathcal{T}_{km}(n)| = \Gamma^{KN-1}(\Gamma+1)^{(K-1)M(KN-1)}. \tag{45}$$

At the receiver side, we need to compute $L'_{k,n}$, the number of independent received directions due to the interference in Y_{kn}. Similar to what is explained for the 3 user case, it is not hard to show that

$$L'_{k,n} = (M-1)L_{k,m}(n) + N\,(\Gamma+1)^{KN-1+(K-1)M(KN-1)}. \tag{46}$$

Therefore, according to Theorem 4, the following number of DoF can be achieved per receive antenna

$$D = \frac{M\,\Gamma^{KN-1}(\Gamma+1)^{(K-1)M(KN-1)}}{1 + M\,\Gamma^{KN-1}(\Gamma+1)^{(K-1)M(KN-1)} + N\,(\Gamma+1)^{KN-1+(K-1)M(KN-1)}}. \tag{47}$$

Taking supremum over Γ, leads to a total number of DoF equal to $\frac{MN}{M+N}K$.

VIII. CONCLUSIONS

In this paper, we obtain new results for the total number of DoF for the constant MIMO interference channel. We show that how a recently introduced interference alignment technique can be uses to achieve higher number of DoF for MIMO interference channel. We also introduce a new outer-bound on the total number of DoF for a MIMO interference channel which coincides with our achievable DoF when the number of users is larger than some threshold which depends on the number of transmit and receive antennas. Since there is no cooperation among transmit/receive antennas of each user in our achievability method, we conclude that in a cellular system as the number of users in each cell increases, the total number of DoF is also increases and approaches to the interference free DoF.

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